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Minimal Orthomodular Lattices

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The lattices called *minimal orthomodular* (MOL) arise in a special exclusion problem concerning the class of all orthomodular lattices (OML) and the subclass of all modular orthocomplemented lattices. This problem was given in G. Kalmbach's book, *Orthomodular Lattices*. We prove that an exclusion system necessarily must contain an infinite lattice. We prove that, except one, all the finite, irreducible MOLs have only blocks with eight elements. We characterize finite MOLs by a covering property related to equational classes generated by the modular ortholattices MO*n*.

1. EXCLUSION PROBLEMS IN LATTICE THEORY

1.1. General Definition

If $C_2 \subset C_1$ are two classes of algebras, an *exclusion system* for $C_2 \subset C_1$ is a subset $\Sigma \subset C_1 - C_2$ such that for $L \in C_1$, $L \notin C_2$ iff there exists $S \in \Sigma$ isomorphic to a subalgebra of L.

Of course, the whole set $C_1 - C_2$ is an exclusion system, but we are interested in exclusion systems as small as possible.

1.2. A Classical Example

We consider the following classes:

C_1: The class of all lattices.
C_2: The class of all modular lattices.
C_3: The class of all distributive lattices.

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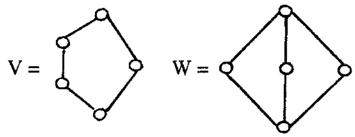
We consider the following lattices:



We have the following classical results (Birkhoff, 1966):

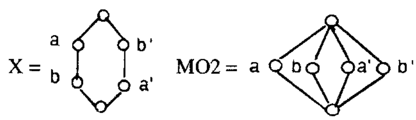
$\{V\}$ is an exclusion system for $C_2 \subset C_1$, $\{W\}$ is an exclusion system for $C_3 \subset C_2$, and $\{V, W\}$ is an exclusion system for $C_3 \subset C_1$.

1.3. Exclusion for Ortholattices

We consider the following classes:

C_1: The class of all ortholattices.
C_2: The class of all orthomodular lattices.
C_3: The class of all modular ortholattices.
C_4: The class of all Boolean algebras.

We consider the following ortholattices:



Denote $U = \{0, 1\}$ the trivial Boolean algebra. We know the following results (Kalmbach, 1982; Carrega and Fort, 1983):

1. $\{X\}$ is an exclusion system for $C_2 \subset C_1$.
2. $\{\text{MO2}, \text{MO2} \times U\}$ is an exclusion system for $C_4 \subset C_2$ and for $C_4 \subset C_3$.

Now we have to study the exclusion problem for $C_3 \subset C_2$.

2. THE EXCLUSION PROBLEM $C_3 \subset C_2$

2.1. Minimal Orthomodular Lattices

Recall that C_2 denotes the class of all OMLs and C_3 denotes the class of all modular ortholattices. Kalmbach (1982, p. 347) gives the following problem: Characterize modular ortholattices L among OMLs by excluding a finite list of *finite* OMLs as subalgebras of L.

Our Theorem 1 gives a negative answer to this problem.

The following result leads to the definition of a minimal orthomodular lattice.

Proposition 1. An OML L belongs (up to isomorphism) to every exclusion system for $C_3 \subset C_2$ iff L is nonmodular and all the proper sub-OMLs of L are modular or isomorphic to L. Such an OML is called a *minimal orthomodular lattice* (MOL).

Theorem 1. Every exclusion system for $C_3 \subset C_2$ must necessarily contain an infinite OML.

For the proof, we use a result in Carrega *et al.* (1990): There exists a nonmodular OML, all finite sub-OMLs of which are modular.

2.2. Finite Minimal Orthomodular Lattices

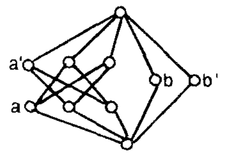
Denote $\mathbb{M}$ the class of all finite MOLs.

Then, $L \in \mathbb{M}$ iff L is a finite nonmodular OML and all the proper sub-OMLs of L are modular.

Proposition 2. If $L \in \mathbb{M}$, then L is irreducible or L is isomorphic to $T \times U$ with $T \in \mathbb{M}$ irreducible.

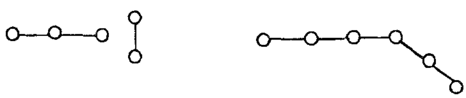
Proposition 3. If $T \in \mathbb{M}$ is irreducible, then $L = T \times U \in \mathbb{M}$.

Example. Denote by T_1 the horizontal sum



T_1 is an irreducible element of $\mathbb{M}$; by Proposition 3, $T_1 \times U$ is an element of $\mathbb{M}$, too.

The Greechie graphs of T_1 and $T_1 \times U$ are

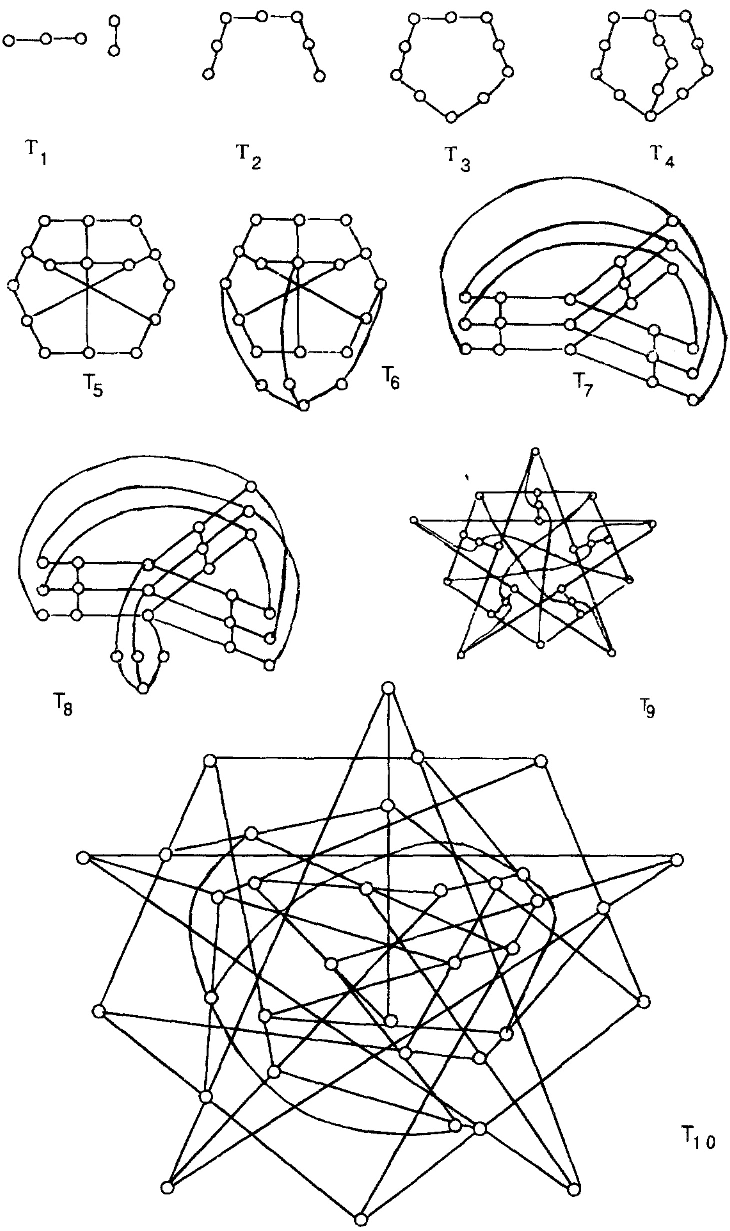


Theorem 2. If $T \in \mathbb{M}$ is irreducible and nonisomorphic to T_1, then every block of T is a Boolean algebra with eight elements.

For the proof we use the following lemma and a result of Bruns and Kalmbach (1972; Kalmbach, 1982, Lemma 6, p. 126).

Lemma. Let $T \in \mathbb{M}$ be irreducible and let $e \in T$, $e \neq 1$; then $[0, e]$ is a modular OML.

We found the following finite, irreducible MOLs, T_i, $1 \leq i \leq 10$ (R. Greechie found T_7 and T_8 and he found the very symmetrical drawing of T_9):



Definition. Let L be a finite OML; we define *distances* between atoms and blocks of L as follows:

- For a and b atoms of L, $a \neq b$, $d(a, b)$ is the number of blocks of a minimal path joining a and b. We complete the definition by $d(a, a) = 0$ and $d(a, b) = \infty$ if there is no path joining a and b.
- For an atom a and a block B, $d(a, B) = \inf d(a, b)$ for a b atom in B.
- For two blocks A and B, $d(A, B) = \inf d(a, b)$ for an a atom in A and a b atom in B.

The following propositions are useful for finding finite MOLs.

Proposition 4. An OML $T \in \mathbb{M}$, irreducible, satisfies the following properties:

1. If $T \neq T_1$, then, for every block B and every atom a, we have $d(a, B) \leq 2$.

2. If $T \neq T_1$ and $T \neq T_2$, then every three consecutive blocks are sides of a pentagon in the Greechie graph of T.

Proposition 5. Let T be a finite, irreducible, nonmodular OML such that:

1. Every block of T is a Boolean algebra with eight elements.

2. For every block B of T and every atom a of T, $d(a, B) \leq 2$.

Then, T is minimal iff for every block B of T and every atom a of T with $d(a, B) = 2$, the sub-OML generated by a and B is T.

Remarks. 1. The characterization of MOLs given in Proposition 5 has been used to built a computer program for testing minimality of finite OMLs.

2. We proved that the only finite, irreducible MOLs satisfying $d(B, B') \leq 1$ for all blocks B and B' are T_2, T_3, T_4, T_5, T_6.

3. Let a be an atom of a finite OML L; the number of blocks B in L with $a \in B$ is called the *degree of a*, denoted $d^\circ(a)$. We prove that the only finite, irreducible MOLs having an atom a with $d^\circ(a) = 1$ are T_1, T_2, T_3, T_4, T_6, T_8.

3. THE COVERING PROPERTY

If T is an OML, $[T]$ denotes the equational class generated by T.

Theorem 3. 1. Let T be a finite, nonmodular, irreducible OML; then T is minimal iff there exists $n \geq 2$ such that $[T]$ covers $[\mathrm{MO}n]$.

2. If T and T' are two finite, irreducible, minimal OMLs that are nonisomorphic, then the classes $[T]$ and $[T']$ are incomparable.

For the proof, we use a result of Jonsson (1967).

Remarks. 1. $[T_1]$, $[T_2]$, $[T_3]$ cover [MO2] and there is no other finite, irreducible MOL satisfying this property (Bruns and Kalmbach, 1972; Kalmbach, 1982, p. 121).

2. $[T_4]$, $[T_5]$, $[T_6]$, $[T_7]$, cover [MO3].

3. $[T_8]$ covers [MO4].

4. $[T_9]$ and $[T_{10}]$ cover [MO5].

5. For every $n \geq 2$ there are finitely many irreducible MOLs T such that $[T]$ covers [MOn].

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